Optimal hierarchical system of road networks
with inward, outward, and through traffic

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This paper formulates a continuous model of a grid road network for determining the optimal
hierarchical system of road networks. The hierarchical system is represented as the proportions
of area taken up by roads at each level of the hierarchy. The objective is to derive the optimal
ratio of road areas that minimizes the total travel time. The model explicitly takes into account
inward, outward, and through traffic as well as inner traffic to examine how traffic composition
affects the optimal ratio of road areas. The result reveals that the optimal ratio of major arterial
roads increases with inward, outward and through traffic.

Keywords: Grid road network; Road area; Traffic composition; Total travel time
Introduction

Road network planning involves a hierarchy of network links, ranging from wide high-speed roads to narrow low-speed roads. On major arterial roads, access to roadside facilities is restricted so that much traffic flows smoothly. On access roads, in contrast, the traffic volume and travel speed are strictly regulated. For efficient road networks, an appropriate hierarchical system must be established, as first discussed by Buchanan (1963).

Some of the works concerning a hierarchical network design have been based on discrete network models, in which demand occurs only at nodes of a network (see, e.g., Current et al. 1986; Current 1988; Pirkul et al. 1991; Balakrishman et al. 1994; Chopra and Tsai 2002). The focus of these discrete models is on developing algorithms and obtaining numerical solutions. Discrete models are to be contrasted with continuous models, in which demand occurs anywhere on a plane. Continuous models, which we address in this paper, often yield simple closed form solutions, leading to a better understanding of fundamental relationships between variables. In continuous models, idealized networks such as a grid network have been used as an approximation of actual networks. Creighton et al. (1960) found the efficient spacings for a square grid network with three road types by minimizing the sum of travel and construction costs. Tanner (1968) evaluated the network of parallel motorways and of rectangular grid motorways from the average travel time. Fawaz and Newel (1976a,b) obtained the optimal spacings for two, three, and four level rectangular grid networks. Chien and Schonfeld (1997) determined the optimal route spacings and operating headways for a grid transit system. Aldaihani et al. (2004) developed a model for designing a hybrid grid network with on-demand vehicles and fixed bus lines. Miyagawa (2009) derived the optimal hierarchical systems that minimize the average and maximum travel time for a grid road network. Continuous models have also been used for transportation planning (Vaughan, 1987), freight distribution problems (Langevin et al., 1996), and facility location problems (Plastria, 1995).

This paper formulates a continuous model of a grid road network for determining the optimal hierarchical system of road networks. The hierarchical system is represented as the proportions of area taken up by roads at each level of the hierarchy. Since the area devoted to all roads in a city is limited, an efficient allocation among different types of roads is necessary. The most related paper by Miyagawa (2009) considered only inner traffic, that is, both origin and destination were
assumed to be within the city. The present paper extends the scope by incorporating inward, outward, and through traffic. This extension allows us to examine how traffic composition affects the optimal hierarchical system of road networks. Analysing the effect of traffic composition is important, because travel time depends on traffic composition. For example, the travel time on major arterial roads increases with the volume of through traffic, which mainly uses major arterial roads. Hence, if the proportion of through traffic is high, much major arterial roads should be constructed to reduce travel time. Different composition will lead to different solutions.

The rest of this paper is organized as follows. The next section formulates a grid network model. Section 2 finds the hierarchical system that minimizes the total travel time. The effect of traffic composition on the optimal hierarchical system is then examined. Section 3 presents a numerical example together with the hierarchical system of actual road networks. The final section concludes with several results.

1 Grid network model

Consider a square city with side length $A$, as shown in Fig. 1. The city has a grid road network. Roads are classified into three types: access roads, minor arterial roads, and major arterial roads. Access roads exist everywhere, whereas minor and major arterial roads form $n_1 \times n_1$ and $n_2 \times n_2$ grid networks, respectively. Let $a_1, a_2$ ($a_1 \leq a_2$) be the spacings and $\Lambda_1, \Lambda_2$ ($\Lambda_1 \geq \Lambda_2$) be the total lengths of minor and major arterial roads, i.e., $\Lambda_1 = 2n_1 \cdot A = 2A^2/a_1$, $\Lambda_2 = 2n_2 \cdot A = 2A^2/a_2$. Then the spacings $a_1, a_2$ are expressed in terms of the road lengths $\Lambda_1, \Lambda_2$ as

$$a_i = \frac{2A^2}{\Lambda_i}, \quad (i = 1, 2).$$

Let $w_1, w_2$ ($w_1 \leq w_2$) be the widths of minor and major arterial roads, respectively. We call the product of the road width and the road length the road area. Let $S_1, S_2$ be the total road areas of minor and major arterial roads, respectively. The objective is to find the ratio of road areas $S_2/S_1$ that minimizes the total travel time.
Origins and destinations are assumed to be uniformly and independently distributed throughout the study region that includes the city. Every traveller moves from origin to destination with frequency normalized at unity, irrespective of its travel distance. Although this assumption is invalid in practice, the result based on such simple situation allows us to obtain fundamental characteristics of road networks that are independent of specific situations. In fact, the uniform demand assumption has been frequently used in continuous transportation models. The result will also supply building blocks for further analyses with non-uniform demand. For example, travel demand that depends on the trip length can be incorporated by using the distribution of trip lengths, as in Creighton et al. (1960); Fawaz and Newel (1976a). Creighton et al. (1960) used an approximation of empirical data, whereas Fawaz and Newel (1976a) used an exponential distribution. More generalized travel demand can be described with spatial interaction models (see, e.g., Taaffe et al. 1996).

Every traveller has to use both minor and major arterial roads as well as access roads. Transfers to different level roads are only allowed at intersections. The movement of a traveller is depicted in Fig. 2. First, the traveller moves from origin along access roads to the nearest intersection of minor arterial roads (Fig. 2(a)). At the intersection, s/he transfers to minor arterial roads and moves to the nearest intersection of major arterial roads (Fig. 2(b)). If the intersection of minor arterial roads coincides with that of major arterial roads, the travel time on minor arterial roads is regarded as zero. Then s/he transfers to major arterial roads and moves to the intersection nearest to destination (Fig. 2(c)). Using again minor arterial roads and access roads, s/he arrives at destination. Obviously, this routing, which we call the nearest intersection routing, will not necessarily provide the minimum travel time route. Employing the minimum travel time routing, however, makes the analysis analytically intractable, because whether or not travellers should detour from the shortest distance route to a higher speed road depends on the locations of origin and destination, as shown in Campbell (1992). Thus we assume the above nearest intersection routing to obtain a simple approximation of travel time. The difference in travel time between the nearest intersection routing and the minimum travel time routing was examined for actual road networks in Miyagawa (2009).

<< Figure 2 >>
Traffic in the city is divided into four groups according to origin and destination. If both origin and destination are inside the city, traffic is called inner traffic. If origin (destination) is outside the city and destination (origin) is inside the city, it is called inward (outward) traffic. If both origin and destination are outside the city, it is called through traffic.

2 Optimal ratio of road areas

To obtain a simple expression of travel time, we use a continuous approximation. That is, the distance between two points on the same level road is measured as rectilinear distances on a continuous plane. Then the distance $R$ between two points $(x_1, y_1), (x_2, y_2)$ is given by

$$R = |x_1 - x_2| + |y_1 - y_2|. \tag{2}$$

Consider rectilinear distances in a square with side length $a$. Let $R_a$ be the average distance from any randomly chosen point in the square to the centre (Fig. 3(a)), $R_b$ be the average distance between any two randomly chosen points in the square (Fig. 3(b)), $R_c$ be the average distance from any randomly chosen point in the square to a side (Fig. 3(c)), and $R_d$ be the average distance between any two randomly chosen sides (Fig. 3(d)). Under the rectilinear distance metric (2), $R_a, R_b, R_c, R_d$ are

$$R_a = \frac{a}{2}, \tag{3}$$
$$R_b = \frac{2}{3}a, \tag{4}$$
$$R_c = \frac{a}{2}, \tag{5}$$
$$R_d = a, \tag{6}$$

respectively (Fairthorne, 1963).

Using these average distances, we calculate the average travel time on each level road. Let $v_0, v_1, v_2$ ($v_0 \leq v_1 \leq v_2$) be the travel speeds on access roads, minor arterial roads, and major arterial roads, respectively. Travels on access roads (Fig. 2(a)) are approximated by travels to the
centre of the square with side length $a_1$. Then from (3), the average travel time on access roads $T_0$ is

$$T_0 = \frac{a_1}{2v_0} = \frac{A^2}{v_0 \lambda_1}. \tag{7}$$

Similarly, travels on minor arterial roads (Fig. 2(b)) are approximated by travels to the centre of the square with side length $a_2$. Then the average travel time on minor arterial roads $T_1$ is

$$T_1 = \frac{a_2}{2v_1} = \frac{A^2}{v_1 \lambda_2}. \tag{8}$$

Travels on major arterial roads (Fig. 2(c)) are approximated by travels in the square with side length $A$. Travels of inner, inward/outward, and through traffic are approximated by travels between any two randomly chosen points in the square, travels between any randomly chosen point in the square and a side, and travels between any two randomly chosen sides, respectively. Then from (4)–(6), the average travel time on major arterial roads $T_2$ is

$$T_2 = \begin{cases} 
\frac{2A}{3v_2} & \text{(inner traffic)} \\
\frac{A}{2v_2} & \text{(inward/outward traffic)} \\
\frac{A}{v_2} & \text{(through traffic)}
\end{cases} \tag{9}$$

The accuracy of the above continuous approximation was examined for inner traffic by Miyagawa (2009) who demonstrated that the approximation becomes better as the road lengths at each level increase.

The total travel time in the city is obtained as follows. Let $q$ be the total traffic volume, and $\alpha, \beta, \gamma$ be the proportions of inner, inward/outward, and through traffic, respectively. Inner traffic uses access roads and minor arterial roads twice, and major arterial roads once, as depicted in Fig. 2. Then the total travel time of inner traffic is

$$\alpha q (2T_0 + 2T_1 + T_2) = \alpha q \left( \frac{2A^2}{v_0 \lambda_1} + \frac{2A^2}{v_1 \lambda_2} + \frac{2A}{3v_2} \right). \tag{10}$$
Inward and outward traffic use access roads, minor arterial roads, and major arterial roads once, because either origin or destination is outside the city. Then the total travel time of inward/outward traffic is

$$\beta q(T_0 + T_1 + T_2) = \beta q \left( \frac{A^2}{v_0 \Lambda_1} + \frac{A^2}{v_1 \Lambda_2} + \frac{A}{2v_2} \right). \quad (11)$$

Through traffic uses only major arterial roads. Then the total travel time of through traffic is

$$\gamma q T_2 = \gamma q \frac{A}{v_2}. \quad (12)$$

The travel speeds on access roads $v_0$ and minor arterial roads $v_1$ are assumed to be constant irrespective of traffic volume. For the travel speed on major arterial roads $v_2$, we assume, as a simple example,

$$v_2 = k \frac{S_2}{q} \quad (13)$$

where $k$ is a constant. This means that the travel speed is proportional to the road area $S_2$, and inversely proportional to the traffic volume $q$. More realistic relationships between travel speed, traffic volume, and traffic density have been proposed in the field of transportation engineering. The choice of (13), however, makes little impact on the basic properties of the result, because our model depends on the continuous approximation. In addition, by using (13), we can derive a closed form solution.

The problem to find the road lengths $\Lambda_1, \Lambda_2$ that minimize the total travel time $T$ is formulated as follows:

$$\min \quad T = (2\alpha + \beta) \frac{qA^2}{v_0 \Lambda_1} + (2\alpha + \beta) \frac{qA^2}{v_1 \Lambda_2} + (4\alpha + 3\beta + 6\gamma) \frac{q^2A}{6kw_2 \Lambda_2}$$

s.t. 

$$w_1 \Lambda_1 + w_2 \Lambda_2 \leq S$$

$$\Lambda_1 \geq \Lambda_2 \geq 0. \quad (14)$$

The objective function minimizes the sum of the travel time on access roads, minor arterial roads, and major arterial roads. The first constraint ensures that the total road area is less than or equal to a constant $S$.  

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Solving (14) yields the optimal ratio of road lengths $\Lambda^*_2/\Lambda^*_1$ as

$$\frac{\Lambda^*_2}{\Lambda^*_1} = \sqrt{\frac{v_0 w_1}{v_1 w_2} \left( 1 + \frac{(4\alpha + 3\beta + 6\gamma)v_1 q}{6(2\alpha + \beta)kw_2 A} \right)}.$$  

(15)

Then the optimal ratio of road areas $S^*_2/S^*_1$ is

$$\frac{S^*_2}{S^*_1} = \frac{w_2 \Lambda^*_2}{w_1 \Lambda^*_1} = \sqrt{\frac{v_0 w_2}{v_1 w_1} \left( 1 + \frac{(4\alpha + 3\beta + 6\gamma)v_1 q}{6(2\alpha + \beta)kw_2 A} \right)}.$$  

(16)

It can be seen that the optimal ratio $S^*_2/S^*_1$ increases with the total traffic volume $q$. For the effect of traffic composition, Fig. 4 shows the optimal ratio as a function of proportions of inward/outward traffic $\beta$ and through traffic $\gamma$. The parameters are as follows: $A = 5$ [km], $v_0 = 10$ [km/h], $v_1 = 20$ [km/h], $w_1 = 4$ [m], $w_2 = 16$ [m], $q = 1 \times 10^5$, $k = 2 \times 10^6$. Note that the higher the proportions of inward/outward and through traffic, the more major roads are required to minimize the total travel time. If all the traffic is through traffic, i.e., $\gamma = 1$, all roads become major arterial roads, and the optimal ratio diverges to infinity.

<< Figure 4 >>

3 Numerical example

Consider a city of $N^2$ square zones, as shown in Fig. 5. Each zone has a grid road network. Let us calculate the optimal ratio of road areas for each zone. Origins and destinations are assumed to be uniformly and independently distributed throughout the city. Travellers of interzonal traffic minimize the number of turns in choosing the shortest route. If there exist two shortest routes with an equal number of turns, the traffic volume is divided between the two routes.

<< Figure 5 >>
Let $A$ be the side length of each zone. The volume of inner, inward, outward, and through traffic of zone $(i, j)$, denoted by $q_{ij}^I$, $q_{ij}^{II}$, $q_{ij}^{III}$, $q_{ij}^{IV}$, are

\begin{align}
q_{ij}^I &= A^4, \\
q_{ij}^{II} &= q_{ij}^{III} = A^4(N^2 - 1), \\
q_{ij}^{IV} &= A^4 \left\{ 2(i - 1)(N - i)N + 2(j - 1)(N - j)N \right. \\
&\quad \left. + (j - 1)(N - 1) + (N - j)(N - 1) \right\},
\end{align}

respectively. The volume of inner, inward, and outward traffic is identical for all zones, whereas the volume of through traffic has its maximum at $i = j = N/2$. This implies that much through traffic passes the city centre. The total traffic volume $q$, and the proportions of inner, inward/outward, and through traffic $\alpha$, $\beta$, $\gamma$ are

\begin{align}
q &= q_{ij}^I + q_{ij}^{II} + q_{ij}^{III} + q_{ij}^{IV}, \\
\alpha &= q_{ij}^I / q, \\
\beta &= (q_{ij}^{II} + q_{ij}^{III}) / q, \\
\gamma &= q_{ij}^{IV} / q,
\end{align}

respectively. Substituting (20)–(23) into (16) yields the optimal ratio of road areas $S_2^* / S_1^*$, as shown in Fig. 6, where $N = 5$ and other parameters are the same as those of Fig. 4. The optimal ratio has a maximum at the city centre and decreases with the distance from the centre. This makes sense intuitively, because much through traffic requires much major arterial roads. Although the result varies according to parameters, the dome-like shape of the optimal ratio remains the same.

This result is also consistent with the hierarchical system of actual road networks. Figure 7 shows the ratio of road areas $S_2 / S_1$ for 23 special wards of Tokyo. We call roads the width of which is more than or equal to 5.5 [m] major arterial roads, and less than 5.5 [m] minor arterial roads. This is because roads with width 5.5 [m] or more usually have two or more lanes. Note that the ratio is high in the central part of Tokyo. The three highest ratios are those of central wards such
as Chiyoda (2.5), Koto (2.2), and Chuo (2.1).

<< Figure 7 >>

4 Conclusion

This paper has provided a useful guideline for designing a hierarchical system of road networks. We have developed a simple grid network model that explicitly takes into account inward, outward, and through traffic as well as inner traffic, thereby examining how traffic composition affects the optimal hierarchical system. The optimal ratio of major arterial roads increases with inward, outward and through traffic. Thus, the large city that attracts much traffic requires more major arterial roads than the small city where inner traffic is dominant. If origins and destinations are uniformly distributed throughout a city, the optimal ratio of major arterial roads has a maximum at the city centre. It follows that if travel demand is not uniform but concentrated at the city centre, more major arterial roads should be constructed at the centre.

Our model deals with the overall level of service in that we minimize the total travel time. Adopting equity-oriented objective functions such as the maximum travel time would be a desirable extension. Relaxing the uniform demand assumption is another direction for future research.

References


J. Tanner, A theoretical model for the design of a motorway system, *Transportation Research* 2, 123–141 (1968).

Figure 1: Square city with a grid road network.

Figure 2: Movement of travellers.
Figure 3: Rectilinear distance in a square.

Figure 4: Optimal ratio of road areas.

Figure 5: City of $N^2$ square zones.
Figure 6: Optimal ratio of road areas.

Figure 7: Ratio of road areas in Tokyo.